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LETTER TO THE EDITOR

## A mean-field approximation for the maximum Lyapunov exponent of Bernoulli trials of matrices

G Paladin<sup>†</sup> and A Vulpiani<sup>‡</sup>

<sup>†</sup> Laboratoire de Physique Théorique de l'Ecole Normale Supérieure<sup>§</sup>, 24 rue Lhomond, 75231 Paris Cedex 05, France and GNSM-CISM Unità di Roma, Italy

<sup>‡</sup> Dipartimento di Fisica dell'Università 'La Sapienza', Piazzale le Aldo Moro 2, I-00185 Roma, Italy and GNSM-CISM Unità di Roma, Italy

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**Abstract.** We consider the infinite products of  $d \times d$  random matrices  $X_i$  which are taken according to a Bernoulli distribution (i.e.  $X_i = A$  with probability  $(1-p)$  and  $X_i = B$  with probability  $p$ ). The maximum Lyapunov exponent is estimated by selecting a suitable periodic sequence. We can thus limit ourselves to compute the eigenvalues of one particular matrix which is obtained as a *finite* product of matrices  $A$  and  $B$ . We also discuss the cases in which the method fails because of fluctuation effects.

The product of random matrices has a great importance in the study of disordered systems (see, e.g., Derrida and Gardner 1984, Paladin and Vulpiani 1987) as well as in the theory of dynamical systems (Benettin 1984, Paladin and Vulpiani 1986, Livi *et al* 1987a, b). The calculation of the Lyapunov characteristic exponents, LCE, is one of the main problems that arises when considering these products.

The knowledge of the LCE in fact allows one to obtain the physical quantities (e.g. free energy and correlation length in statistical mechanics system (De Calan *et al* 1985), localisation length and density of state in the localisation problem (Derrida and Gardner 1984)) via the transfer matrix method. On the other hand, the maximal LCE measures the global degree of chaoticity in dynamical systems since it gives an estimate of the typical time on which the memory of the initial conditions is loose (Benettin 1984, Paladin and Vulpiani 1986, Livi *et al* 1987a, b).

In this letter, we consider the product of real  $d \times d$  random matrices  $X_i$  which are Bernoulli distributed, i.e.  $X_i = A$  with probability  $(1-p)$  and  $X_i = B$  with probability  $p$ . The maximal LCE  $\gamma(p)$  is defined as

$$\gamma(p) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \frac{\|M_N \mathbf{z}(0)\|}{\|\mathbf{z}(0)\|} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln |\text{Tr } M_N| \quad (1)$$

where

$$M_N = \prod_{i=1}^N X_i \quad (2)$$

and  $\mathbf{z}(0) \in \mathbb{R}^D$  is a generical vector.

It can be proved that the limit (1) exists for almost all matrices  $M_N$  given by (2) and does not depend on  $\mathbf{z}(0)$  with probability 1 (Oseledec 1968).

<sup>§</sup> Laboratoire Propre du Centre National de la Recherche Scientifique, associé à l'Ecole Normale Supérieure et à l'Université de Paris-Sud.

Our purpose is to estimate  $\gamma(p)$  by a simple method which takes account of the non-commutative nature of our 'coin-tossing' process. A naive approximation, in analogy with the scalar case, is

$$\gamma(p) = (1-p)\gamma_A + p\gamma_B \quad (3)$$

where  $\gamma_A = \gamma(p=0)$  and  $\gamma_B = \gamma(p=1)$  are the logarithms of the modulus of the largest eigenvalue of  $A$  and  $B$ . Note that  $[A, B] = 0$  is not sufficient for (3) to be valid; one also needs the eigenvectors of  $A$  and  $B$  corresponding to the most expansive directions *not* to be orthogonal. On the other hand, there are no generical methods to find  $\gamma(p)$ , even if perturbative expansions in  $p$  can work in a lot of particular cases.

Let us now remark that in the limit of large  $N$  the number  $n_A$  of matrices  $A$  in the product  $M_N$  is given by the large number law:

$$n_A = (1-p)N + O(N^{1/2}). \quad (4)$$

This means that among all the possible sequences  $\{X_i\}_{i=1, \dots, N}$  only the class of those with  $n_A = (1-p)N$  gives a relevant contribution to the calculation of  $\gamma(p)$  while the class of remaining sequences has vanishing probability in the limit  $N \rightarrow \infty$ . This observation does not simplify our problem too much but indicates how to pick up a typical sequence. If we neglect the fluctuation role, we can try to look for a periodic sequence in the first relevant class, so that the calculation of  $\gamma(p)$  is reduced to the calculation of the maximal eigenvalue of a particular matrix. Let us choose for simplicity  $p = 1/(n+1)$ , with  $n = 1, 2, \dots$ . In this case, the smallest period length for which  $n_A = N(1-p)$  is  $n+1$  and the 'mean-field' typical sequence will be

$$\{X_i\}_{i=1, (n+1)N} = (A^n B)^N. \quad (5)$$

It is quite easy to find a generalisation of the sequence (5) if  $p = m/[(n+1)m+1]$  with  $n$  and  $m$  integer. One obtains

$$\{X_i\}_{i=1, N[(n+1)m+1]} = [(A^n B)^m A]^N.$$

In the following we shall, however, only consider sequences of the kind (5).

Let us also note that the maximal LCE of sequences (5) is invariant under cyclic permutations  $(A^{n-i} B A^i)$ , with  $i = 0, 1, \dots, n$  even if the whole spectrum of LCE cannot be invariant for  $d > 2$ .

Nevertheless the minimal LCE of the sequences (5) is also invariant under cyclic permutations, since when considering the product of matrices  $A^{-1}$  and  $B^{-1}$  it, of course, becomes the maximum LCE. It follows that all our arguments will be valid for the minimal LCE as well as the maximal one if  $\det A$  and  $\det B$  do not vanish.

Now let us estimate  $\gamma(p)$  by the sequence (5):

$$\lambda_1 \left( p = \frac{1}{n+1} \right) = \frac{1}{(n+1)} \ln |l_1(A^n B)| \quad (6)$$

where  $l_1(A^n B)$  is the largest eigenvalue of  $A^n B$ . Of course, for  $p > 0.5$  we have to consider sequences of the form  $(B^n A)^N$ . Let us stress that the approximation (6)

becomes exact (i.e.  $\gamma = \lambda_1$ ) if  $[A, B] = 0$ . In principle it is possible to improve (6) by a second-order approximation  $\lambda_2(p)$  given by the  $2(n+1)$  length products:

$$\lambda_2\left(p = \frac{1}{n+1}\right) = \frac{1}{2(n+1)(2n+1)} \sum_{i=-n}^n \ln |l_i(A^{n-i}BA^{n+i}B)|. \tag{7}$$

In (7), we have to compute the maximum eigenvalues of  $n+1$  matrices and further approximations of order  $K$  become very complicated for  $n > 2$  (i.e.  $p < 0.25$ ). This systematic way to obtain  $\gamma(p)$  is not our goal, since we do not want to find a good trick for performing perturbative expansions in  $p$ , but just a simple way to give a rough estimate of  $\gamma(p)$ . Let us therefore begin to discuss our approximation for  $2 \times 2$  matrices. We shall consider  $p < 0.5$  and a reference frame where the matrix  $A$  is diagonal without loss of generality. We thus get three cases: real non-degenerate eigenvalues  $l_1 > l_2$ , real degenerate eigenvalues  $l_1 = l_2$  and complex conjugate eigenvalues  $l_1 = l_2^*$ .

(i) Non-degenerate real eigenvalues  $l_1 > l_2$ . This case contains all the one-dimensional statistical mechanics with ‘Bernoulli’ disorder and finite-range interactions. In fact, a theorem of Frobenius (Gantmacher 1964) ensures that the maximum eigenvalue of a finite  $d \times d$  matrix with positive elements is strictly non-degenerate. We have, for instance, considered the Ising model with a site random field  $h_i = h$  with probability  $p$  and  $h_i = 0$  otherwise. The calculation of the partition function can be done via the calculation of the maximum LCE of the random product of two transfer matrices (De Calan *et al* 1985). The agreement between the numerical estimates of  $\gamma(p)$  and  $\lambda_1(p)$  is rather good for all  $p$  and not too small temperatures (the percentage error is  $\sim 1\%$ ). Moreover the ‘scalar’ estimate (3) also works well.

Now let us consider the most general case:

$$A = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix} \quad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{8}$$

with  $\alpha, \beta, \gamma, \delta \neq 0$ . It is easy to verify that, for large  $n$ , one gets

$$\lambda_1\left(p = \frac{1}{n+1}\right) = (1-p)\gamma_A + p \ln|\alpha| + O\left(\left|\frac{l_2}{l_1}\right|^n\right). \tag{9}$$

This result corresponds to the first-order term of the perturbative expansion in  $p$ . Actually, at second order in  $p$  one has (Derrida 1987)

$$\gamma(p) = (1-p)\gamma_A + p \ln|\alpha| + p^2 \sum_{k=0}^{\infty} \ln \left| 1 + \frac{\beta\gamma}{\alpha^2} \left(\frac{l_2}{l_1}\right)^k \right| + O(p^3). \tag{10}$$

We thus see that the error of our estimate is of order  $O(p^2)$ . Figure 1 shows  $\lambda_1(p)$  and  $\gamma(p)$  for the product of symplectic matrices which are relevant for the study of dynamical systems (Paladin and Vulpiani 1987, Benettin 1984):

$$A = \begin{pmatrix} 1 & 1 \\ a & 1+a \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ b & 1+b \end{pmatrix} \tag{11}$$

with  $a, b \notin [-2, 0]$  in order to have real non-degenerate eigenvalues. Let us note that, for  $a = 0.8$  and  $b = -4.9$ ,  $\lambda_1(p = 0.5)$  improves the first-order expansion in  $p$  by about 25% while, for  $a = 5$  and  $b = -4.9$ ,  $\lambda_1(p)$  corresponds to the first-order term of the  $p$  expansion, which however is quite a good estimate of  $\gamma(p)$  for all  $p$ .

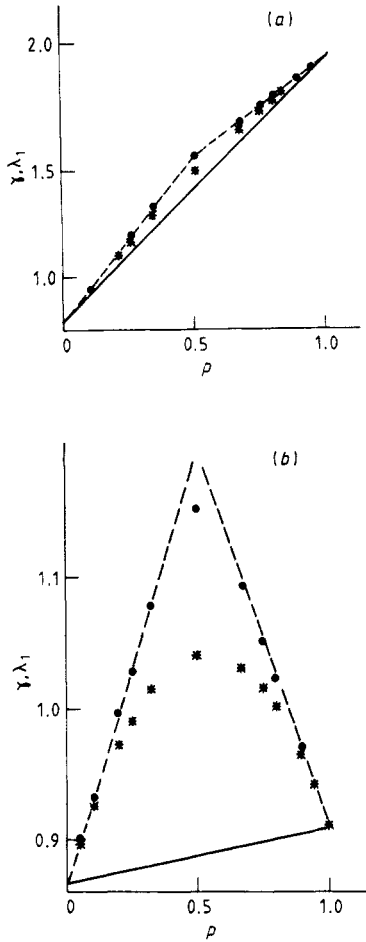


Figure 1.  $\gamma^*$  and  $\lambda_1$  (●) plotted against  $p$  for matrices of the form (11). (a)  $a = 5, b = -4.9$ . (b)  $a = 0.8, b = -4.9$ . The full line indicates the trivial approximation (3), the broken lines show the first-order term of the perturbative expansion in  $p$  (10).

(ii) Degenerate real eigenvalues  $l_1 = l_2$ . We can always consider a basis where the matrix  $A$  is triangular:

$$A = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} l_1. \tag{12}$$

In particular, let us consider  $\omega = 1$  and a matrix  $B$  of the form

$$B = \begin{pmatrix} 1 & 1 \\ b & 1+b \end{pmatrix} \quad b > 0 \tag{13}$$

for recovering the product of symplectic matrices (11) with  $a = 0$ .

All our arguments can, however, be generalised to  $\omega \neq 1$ .

An interesting case is obtained for  $|l_1| = 1$  where one has

$$\lambda_1(p) = (bp)^{1/2} + O(bp). \tag{14}$$

On the other hand, the 'scalar estimate' (3) would give a  $LCE \approx p\sqrt{b}$ . The difference is made evident in figure 2. Let us note that our approximation therefore holds even if  $d\gamma/dp|_{p=0} = \infty$  so that the standard perturbative expansion in  $p$  (10) fails.

Actually, for the products of  $A$  and  $B$  given by (12) and (13) it can be shown (Parisi and Vulpiani 1986) that  $\lambda_1(p)$  is equal to the generalised Lyapunov exponent  $L(1)$  defined as (Benzi *et al* 1985)

$$L(1) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \left\langle \frac{\|M_N z(0)\|}{\|z(0)\|} \right\rangle \tag{15}$$

where  $\langle \rangle$  is the average over the different realisations of the Bernoulli trials. Indeed, in general,  $L(1) \geq \gamma$  and  $L(1) = \gamma$  only if fluctuations of the Lyapunov exponent computed on finite  $N$  products around  $\gamma$  are negligible.

(iii) Complex conjugate eigenvalues. In this case our method may be useless since it neglects the fluctuations. Indeed, fluctuations could sometimes become so relevant that periodical mean-field estimates make no sense. As a typical example let us consider matrices of the form:

$$X_i = \begin{pmatrix} E - V_i & -1 \\ 1 & 0 \end{pmatrix} \tag{16}$$

where  $V_i = V_A$  with probability  $(1-p)$  and  $V_i = V_B$  with probability  $p$ . The product  $M_N$  is thus related to the Schrödinger equation on a one-dimensional lattice:  $\psi_{i+1} + \psi_{i-1} = (E - V_i)\psi_i$  written in a recursive form.

The maximum LCE is the inverse of the localisation length of the eigenfunction  $\{\psi_1, \psi_2, \dots, \psi_N\}$ , i.e.  $|\psi_i| \sim |\psi_0| \exp(-\gamma|i|)$  where  $\psi_0$  is the maximum of the wavefunction and  $|i|$  is large enough.

The eigenvalues of (16) are not real for  $|E - V_i| < 2$  and let us note that for pure systems (i.e.  $p = 0$  or  $1$ ,  $V_i = V$  constant) the energy band is just  $|E - V| \leq 2$ . It is well known that the eigenstates are always localised if  $p \neq 0$  or  $p \neq 1$  while they are extended

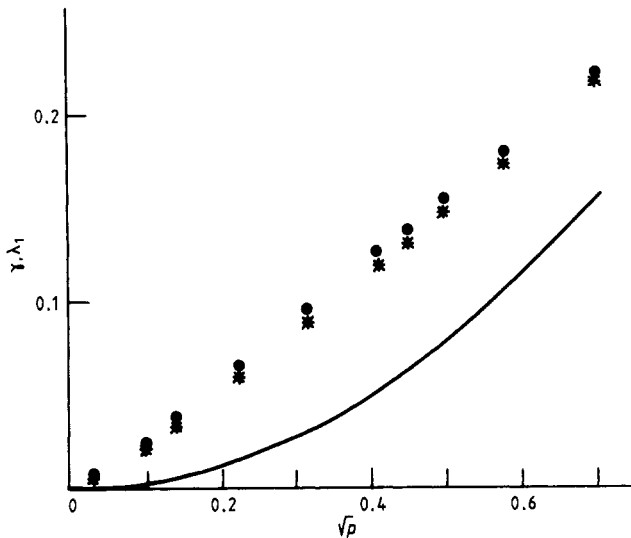


Figure 2.  $\gamma$ (\*) and  $\lambda_1$ (●) plotted against  $\sqrt{p}$  for matrices (11) with  $a = 0$ ,  $b = 0.1$ ; the full curve indicates the trivial approximation (3).

(i.e.  $\gamma = 0$ ) for any periodical potential (Ishii 1973, Carmona *et al* 1987). It follows that  $\lambda_1(p) = (n+1)^{-1} \ln |I_1(A^n B)|$  vanishes for energies inside the band of the pure system. On the other hand,  $\gamma(p) > 0$  since disorder localises the wavefunction. It is therefore evident that any periodical approximation has to fail. The same kind of problem is present for the matrices (11) with  $a, b \in ]-2, 0[$ . Let us remark that troubles also arise in perturbative calculations (Derrida *et al* 1987, Martinelli and Micheli 1987) of the LCE for energies inside the band. Roughly speaking, we can see the origin of the difficulties in the fact that matrices  $A$  and  $B$  are neither expanding nor contracting. The finite value of  $\gamma(p)$  is fully due to fluctuations. Nevertheless, if at least one of the two matrices is expanding, the approximation  $\lambda_1(p)$  still gives reasonable results, as shown in figure 3.

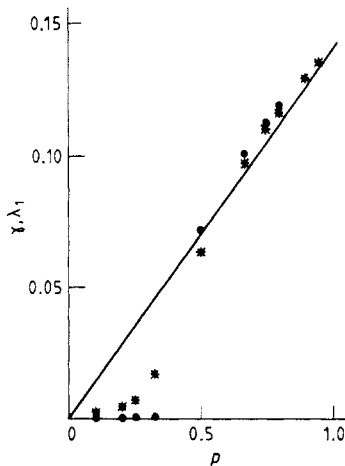


Figure 3.  $\gamma^*$  and  $\lambda_1$  (●) plotted against  $p$  for matrices of the form (16) with  $E = 2$ ,  $V_A = 0.1$ ,  $V_B = -0.2$ ; the full line indicates the trivial approximation (3).

For our purpose  $2 \times 2$  matrices involve all the complexities of  $d \times d$  matrices in practice. Let us, for example, consider a generalisation of (11):

$$A = \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathcal{A} & \mathbb{1} + \mathcal{A} \end{pmatrix} \quad B = \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathcal{B} & \mathbb{1} + \mathcal{B} \end{pmatrix} \quad (17)$$

where  $\mathbb{1}$  is the  $d \times d$  unit matrix and  $\mathcal{A}, \mathcal{B}$  are  $d \times d$  symmetric matrices. If each element of  $\mathcal{A}$  and  $\mathcal{B}$  is non-negative the mean-field approximation (6) works without problem (see figure 4) while if some elements of  $\mathcal{A}$  and/or  $\mathcal{B}$  are negative it might happen that approximation (6) is not satisfactory enough.

Finally, we have investigated the large- $d$  limit in generical matrices without a particular structure in order to see if the fluctuation effects become negligible. In this case our mean-field approximation should be exact for  $d \rightarrow \infty$ . Indeed, we have checked that, already for  $d \geq 4$ ,  $\lambda_1(p)$  is practically equal to  $\gamma(p)$  for matrices of the form:  $A_{ij} = C_A \alpha_{ij}$  and  $B_{ij} = C_B \beta_{ij}$ , where  $\alpha_{ij}$  and  $\beta_{ij}$  are numbers extracted at random in the interval  $[-1, 1]$  according to a uniform probability distribution. Moreover, for these matrices  $\lambda_1(p)$  does not differ by the naive 'scalar' approximation (3). Nevertheless this last feature is not generic: for example, we have found that for *symmetric*  $A$  and  $B$  with elements which are independent Gaussian random variables, (3) fails to estimate  $\gamma(p)$  in the limit  $d \rightarrow \infty$ , while our mean-field approximation (6) becomes very accurate.

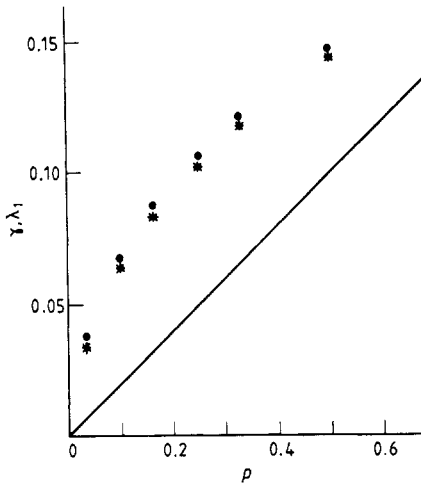


Figure 4.  $\gamma^*$  and  $\lambda_1$  (●) plotted against  $\rho$  for matrices of the form (17) with

$$\mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} 0.05 & 0.01 \\ 0.01 & 0.02 \end{pmatrix}.$$

The full line is the trivial approximation (3).

Let us also remark that the difference between the LCE and the generalised Lyapunov exponent  $L(1)$  defined in (5) vanishes as  $d$  increases, supporting the idea that fluctuations become irrelevant in large 'typical' matrices.

In conclusion we have tried to overcome the difficulties inherent to the non-commutative nature of Bernoulli trials of matrices. The mean-field-like estimate of the maximum LCE is remarkable for its simplicity and because no other general methods exist, as far as we know. It allows us to take account of only one particular periodic sequence, neglecting the order of the trial results in the infinite product of matrices. Moreover we have stressed the cases in which fluctuations become so relevant that any finite-periodic approximation is useless, as in the localisation problem. It is also pointed out that, for matrices without particular structure, fluctuations become negligible and the LCE can be well approximated by our method in the limit of large matrix size.

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